Exam CONTINUUM MECHANICS - 04/06/2008 - 8:30

T1. (i) Starting from the eigenvalue problem $\mathsf{E} \cdot \mathbf{n} = \epsilon \mathbf{n}$ for the (small) strain tensor E of a continuous medium, give the characterization of the following states of deformations of the medium: general deformation, plane deformation, linear deformation, spherical deformation.

(ii) Show that in case of small strain, the first invariant of the strain tensor, namely $I_E = \mathrm{tr}\, \mathsf{E}$, represents the relative change in volume in the neighbourhood of a point of the medium, i.e.

$$\mathrm{tr}\ \mathsf{E} = \frac{dV - dV_0}{dV_0},$$

where dV_0 represents the volume of an elementary parallellepid around the point under consideration.

(iii) Show that the compatibility condition of St. Venant, $\nabla \times (\nabla \times \mathsf{E})^t = 0$, is a necessary and sufficient condition for a given second order tensor E to be admissible as a strain tensor of a continuum (i.e. to be of the form $\mathsf{E} = 1/2(\nabla \mathbf{u} + \mathbf{u}\, \nabla)$ for some vector function $\mathbf{u}$).

T2. Consider Hooke's law $\mathsf{T} = \lambda(\mathrm{tr}\mathsf{E})\mathsf{1} + 2\mu\mathsf{E}$ for a linear isotropic elastic medium with Lamé's constants λ, μ.

(i) Show that Hooke's law implies $\sigma = (3\lambda + 2\mu)\epsilon$ and $\mathsf{T}' = 2\mu\mathsf{E}'$, where $\mathsf{T} = \sigma\mathsf{1} + \mathsf{T}'$ and $\mathsf{E} = \epsilon\mathsf{1} + \mathsf{E}'$ represent the decompositions of the stress and strain tensor, respectively, in their spherical and deviatoric parts, with $\mathrm{tr}\mathsf{T}' = \mathrm{tr}\mathsf{E}' = 0$. Derive the inverted Hooke's law

$$\mathsf{E} = \frac{1+\nu}{E}\mathsf{T} - \frac{\nu}{E}(\mathrm{tr}\mathsf{T})\mathsf{1},$$

where $\nu = \dfrac{\lambda}{2(\mu + \lambda)}$ and $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ is Young's modulus.

(ii) Given the equation of motion of a continuum $\rho\dot{\mathbf{v}} = \mathbf{f} + \nabla \cdot \mathsf{T}$, derive from it in the case of a linear isotropic elastic medium the linearised version of the Navier-Cauchy equation for the displacement vector $\mathbf{u}$ in the neighbourhood of an equilibrium state, treating $\rho \approx \rho_0$ (the equilibrium value of the density) as a material constant.

(iii) Derive the general plane wave solution for the displacement vector in a general (not necessarily isotropic) linear elastic medium in the absence of body forces (i.e. $\mathbf{f} = \mathbf{0}$). Prove that for any given direction there are in general three different wave modes.

O1. (i) For a certain elastic medium with plane deformation, the small strain tensor is given in Cartesian coordinates (x, y, z) by

$$\mathsf{E} = \begin{bmatrix} 2xy + 3y^2 & 1/2(x^2 + 6xy + y^2) & 0 \\ 1/2(x^2 + 6xy + y^2) & 2xy + 3y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(i) Compute the components $u_x(x, y)$ and $u_y(x, y)$ of the displacement vector $\mathbf{u}$ if the points of the x-axis remain fixed (there is no u_z-component and no z-dependence of u_x and u_y), and find the corresponding rotation vector $\mathbf{w}$.

(ii) Sketch the deformation of what originally was the square $ABCD$, with vertices $A(1, 1, 0), B(1, -1, 0), C(-1, -1, 0), D(-1, 1, 0)$.

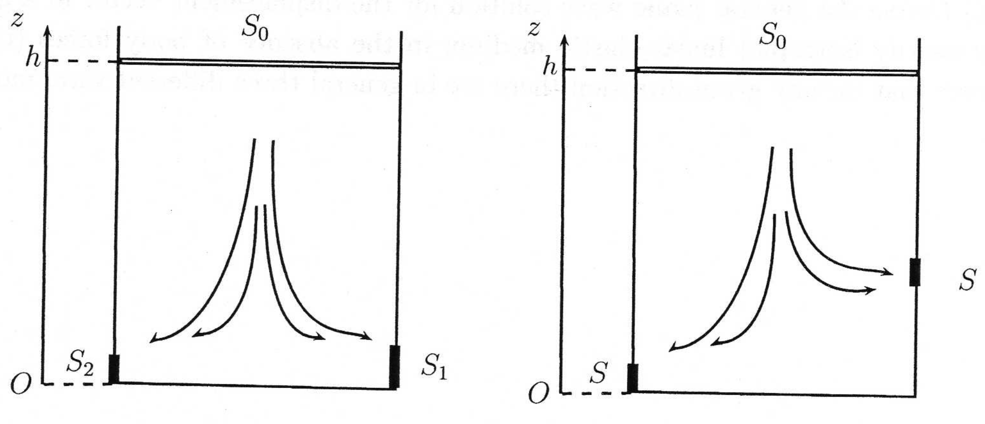
(iii) Determine the stress tensor if the medium satisfies Hooke's law. Find the stress vector $\mathbf{t}^{(n)}$ at the vertex D of the above square associated with the direction $\mathbf{n} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$.

O2. A cilindrical barrel with cross-section S_0 is filled up to a height h with an *ideal, incompressible* fluid. We choose the z-axis vertically upwards, with the bottom level of the barrel at $z = 0$. At the bottom level one then makes two small vertical orifices (holes) in the wall of the barrel with cross-sectional area S_1 and S_2, respectively, through which the liquid can flow out of the barrel (see figure). The flow is *stationary and irrotational*, and we assume there is no atmospheric pressure difference between the upper level and the outflow orifices.

(i) Show that the outflow velocities v_1 and v_2 at the orifices S_1 and S_2, respectively, are equal (i.e. $v_1 = v_2$).

(ii) Compute the instantaneous height $z(t)$ of the liquid in the barrel if the outflow starts at $t = 0$ with $z(0) = h$. After what time (τ) will the barrel be empty?

(iii) What happens if one considers the situation whereby only one outflow orifice, with cross-section S, is located at reference level $z = 0$, whereas the second one, with equal cross section S, is located at height $z = h/4$: what is the relation between the outflow velocities and what is the time τ_1 to empty the barrel in this case?



(1) Theory and exercises must be delivered SEPARATELY.

(2) Don't forget to put your name clearly on top of EVERY sheet.