Algebraïsche en Analytische Getaltheorie
Examen – 23.1.2015

Part I: Analytic number theory (11/20 of the final score)

Instructions: Use the grid paper for your answers. Make sure you write your name on each of the sheets. (If you improve the asked error term in any asymptotic estimate from the problems without using more than what it is allowed in the statement, you will receive extra credit for it!)

Veel succes!

Problem 1

Let

$$S(x) = \sum_{n \le x} \frac{\phi(n!)}{n!}.$$

Obtain an asymptotic formula for $S(x)$ with a simple function of x as main term and error term that is by a factor $O(1/\log x)$ smaller than the main term.

Problem 2

Let $\Psi(x, y)$ be the number of positive integers $n \le x$, all of whose prime factors are $\le y$. Assume that $\sqrt{x} \le y \le x$.

(a) Show that

$$\Psi(x, y) = \lfloor x \rfloor - \sum_{y < p \le x} \left\lfloor \frac{x}{p} \right\rfloor.$$

(b) Without using the Prime Number Theorem (you may use Chebyshev estimates or Mertens estimates), find an appropriate asymptotic formula for $\Psi(x, y)$ in terms of elementary functions of x and y and with error term no worse than $O(x/\log x)$.

Problem 3

Given a positive integer n define its squarefree kernel $k(n)$ by $k(n) = \prod_{p|n} p$. Obtain an asymptotic estimate for $\sum_{n \le x} k(n)/n$ with error term $O(\sqrt{x})$. The constant in the leading term should be evaluated in terms of an infinite product over primes. (You may use, without proof, that $S(x) = \#\left\{n \le x : \ p|n \Rightarrow p^2|n\right\} = O(\sqrt{x})$.)

Problem 4

Let f be a non-negative completely multiplicative arithmetic function. Define

$$g(n) = \begin{cases} g(n) = f(p)\log p & \text{if } n = p^m, \text{where } p \text{ is prime and } m \ge 1, \\ g(n) = 0 & \text{otherwise.} \end{cases}$$

(a) Express the Dirichlet series $G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$ in terms of $F(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$, $F'(s)$, and another Dirichlet series with possibly smaller abscissa of convergence. (*Hint:* The formula should resemble a "perturbation" of that for the Dirichlet series of Λ in terms of the Riemann zeta function)

(b) Assume that $\sum_{n \le x} f(n) = Ax + O(1)$ for some $A > 0$. Prove that g has mean value equal to 1. (*Hint:* You may use a Tauberian theorem, but justify every step in its application.)

(c) Use part (b) to show the following generalization of the Prime Number Theorem: If the asymptotic estimate $\sum_{n \le x} f(n) = Ax + O(1)$ holds for some $A > 0$, then $\sum_{p \le x} f(p) \sim x/\log x$.